Lesson 5
Rotational and Projectile Motion

Introduction: Connecting Your Learning

The previous lesson discussed momentum and energy. This lesson explores rotational and circular motion as well as the particular effects of gravity on those motions. The relationship between angular momentum and torque is investigated, and projectile motion is examined from a qualitative perspective as well as a quantitative, mathematical perspective. The special case of satellite projectiles is discussed along with Kepler's Laws of Planetary Motion.

Readings, Resources, and Assignments	
Required Textbook Readings	*Conceptual Physics* Chapter 1, Section 1.5 Chapter 3 *Newtonian Physics* Chapter 10, Sections 10.1 and 10.2

Check Prior Knowledge

Determine whether each statement below is true or false. (Answers are located at the end of the lesson.)

True or False

1. Objects moving in a circle require a centrifugal force.
2. The gravitational force between two objects depends on the mass of the objects.
3. In order to escape the gravitational force of the Earth, a body must be launched at a specified minimum speed, regardless of the forces acting on the launched body.

Focusing Your Learning

Lesson Objectives

By the end of this lesson, you should be able to:

1. Define uniform circular motion and solve selected problems involving centripetal force.
2. Differentiate between circular motion and rotational motion.

3. Explain the relationship between torque and angular acceleration.

4. Use Newton's Law of Universal Gravitation to find the forces exerted between masses.

5. Describe projectile motion in terms of the kinematics variables, displacement, velocity, acceleration, angle of trajectory and time.

6. Use Kepler's Laws of motion to discuss satellite and planetary motion.

Approaching the Objectives

This lesson is comprised of five sections:

Section 1: Uniform Circular Motion

Section 2: Angular Kinematics

Section 3: Rotational Motion, Torque, and Angular Momentum

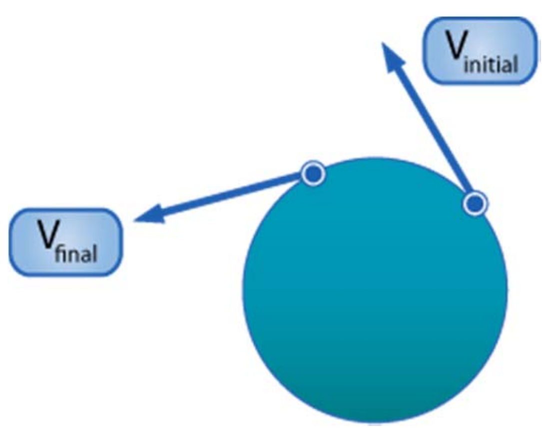
Section 4: Gravity

Section 5: Projectile and Satellite Motion

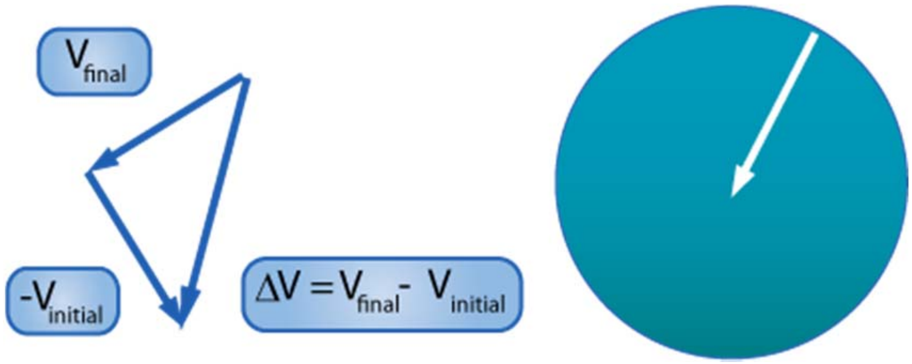
Section 1: Uniform Circular Motion

Begin this lesson by reading Chapter 3, Section 3.1, pp. 63 to 67 in the *Conceptual Physics* textbook. View these videos from Khan Academy on Introduction to Centripetal Acceleration Part 1, Part 2 and Part 3.

When an object moves in a circle (even if the speed is constant), the velocity is not constant because the direction of the velocity is continuously changing. This means the velocity vector is always tangent to the circular path the object is taking. As previously learned, if the velocity is changing, then there must be acceleration. By Newton's 2^{nd} Law, if the object is accelerating, then there must be net force acting on the object in the same direction as the acceleration.



You may recall that acceleration equals the change in velocity divided by the time interval over which that change took place. If the vector velocities are subtracted, $v_{final} - v_{initial}$, the vectors look like the diagram below:



Recall, subtracting vectors is the same as adding the negative of the vector.

When the $v_{final} - v_{initial}$ vector is moved to the circle, you can see the direction of the vector points toward the center of the circle. Regardless of where on the circle the vectors are chosen, the Δv difference vector will always point toward the center of the circle. The conclusion is that the acceleration, and therefore, the force, that cause circular motion must point toward the center of the circular path.

For an object moving in a circle at constant speed, the magnitude of the velocity is the circumference of the circle $(2\pi r)/t$, where r is the radius of the circle and t is the time required to make one trip around the circle. The magnitude of the acceleration is $(2\pi v)/t$. Combining these two equations gives $a_{centripetal} = v^2/r$.

Since Newton's 2nd Law is $F_{net} = ma$, it follows that $F_{centripetal} = m\, v^2/r$.

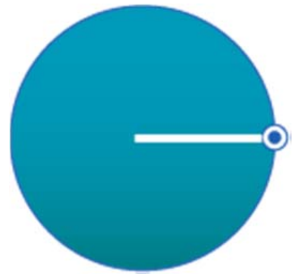
It is important to note that centripetal force is not a separate force that appears on a free body diagram. The centripetal force must be caused by another force or combination of forces, such as the tension in a string, gravitational attraction, or the friction between the tires and the road. If the object is moving in a circle, the net resultant force is called centripetal (center-seeking) force.

Example: A 25 g stone is tied to the end of a string and moves in a horizontal, circular path. The radius of the circle is 1.5 m. The stone makes 25 complete revolutions in 30 s.

Find the tension in the string.

Solution:

Make a sketch:



$F_c = mv^2/r$

$v = 2\pi\ r/T$

$T = 30\ s/25\ rev = 1.2\ s$

$v = (2\pi\ 1.5\ m)/1.2\ s = 7.85\ m/s$

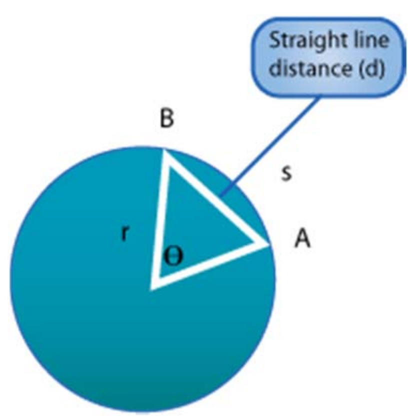
$F_c = mv^2/r = \{(0.025\ kg)(7.85\ m/s)^2\}/1.5\ m = 1.03\ N$

The next section defines angular variables and relates the linear quantities of displacement, velocity, and acceleration to their counterparts, angular displacement (theta, θ), angular velocity, (omega, ω) and angular acceleration (alpha, α).

Section 2: Angular Kinematics

View the video from Kahn Academy on Angular Velocity.

Displacement was defined in previous lessons as the straight line distance between two points. Angular displacement (θ) is defined in terms of an angle measured along a circular arc.



When the arc, s, is equal to the radius, r the angle θ is defined to be one radian. This is equal to 57.3°. The relationship between s and θ is usually written as:

$$s = r\,\Theta$$

Notice that the straight line distance (d) between points A and B in the diagram above is not the same as the arc (s) between A and B. Using methods of advanced mathematics, it is easy to show that as the arc gets smaller and smaller, the straight line distance becomes equivalent to the arc length. The rate at which the angular displacement changes with respect (WRT) to time is defined as the angular velocity (ω).This is related to the linear velocity by the equation:

$$v = r\,\omega$$

The rate at which the angular velocity changes (WRT) time is defined as the angular acceleration (α).This is related to the linear acceleration by the equation:

$$a = r\,\alpha$$

It can be shown that for every linear kinematics equation, there is a corresponding angular kinematics equation. Fill in the table shown below with the angular equation that corresponds to the linear equation given. The answers are at the bottom of the lesson.

Linear	Angular
$s = s_o + v_ot + ½\ at^2$	$\Theta = \Theta_o + \omega_ot + 1/2\ \alpha t^2$
$v = v_o + at$	
$v^2 = v_o^2 + 2as$	
$v_{avg} = (v_o + v_f)/2$	

Practice:

1. How long is an arc subtending 2.31 rad with a radius of curvature of 5 m?
2. A disk spins at an angular velocity of -0.00804 rad/s. If it is an angle of 4.58 rad at time = 0, how much later will it be at an angle of 4.42 rad?
3. If a car accelerates from 34.4 km/h to 82.7 km/h in 12.5 s, and the tires have an angular acceleration of 5.339 rad/s^2, what is the radius of the tires?

Like angular kinematics, there is also correlation between net force (responsible for linear acceleration) and torque (T), the quantity that is responsible for angular acceleration. Angular acceleration depends not only on mass, but on how that mass is distributed about an axis of rotation.

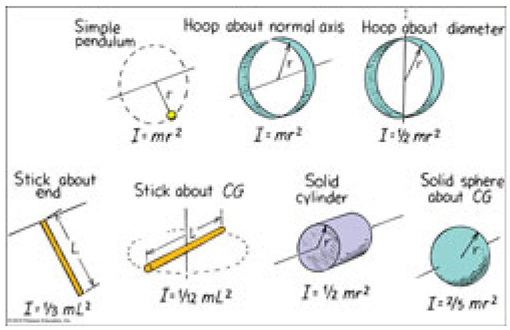
$F_{net} = ma$
$T_{net} = I\,\alpha$

Section 3: Rotation Inertia, Torque, and Angular Momentum

Begin this section by reading Chapter 3, Section 3.2, pp. 68 to 70 in the *Conceptual Physics* textbook.

View the video from Khan Academy on Torque.

Recall the term inertia from previous lessons. Inertia is a property of an object. Mass is a measure of how much inertia an object contains. Mass is not the only parameter that affects motion when an object rotates about an axis. The rotation also depends on how the mass is distributed and where the axis of rotation is located. The rotational inertia (I) is related to the mass, in addition to the distance from the object to the axis of rotation. The mathematics to calculate these quantities requires knowledge of calculus, so these formulas are given and need not be memorized. The figures from the text are shown here.



Example:

Which object is easier to rotate, a solid cylinder or a solid sphere (assume the mass and radius of both are the same)?

For a solid cylinder, $I = ½\ mr^2$ and for a solid sphere, $I = 2/5\ mr^2$. Since $2/5 < ½$, the rotational inertia for the sphere is less than the rotational inertia for the cylinder. Therefore, it is easier to rotate the sphere.

For linear motion, Newton's 2nd Law relates the net force to the linear acceleration. The rotational counterpart to net force is net torque and the rotational counterpart to mass is rotational inertia.

The net torque acting on an object tends to produce rotation about an axis. The angular acceleration produced is directly proportional the net torque and inversely proportional to the rotational inertia.

Section 4: Gravity

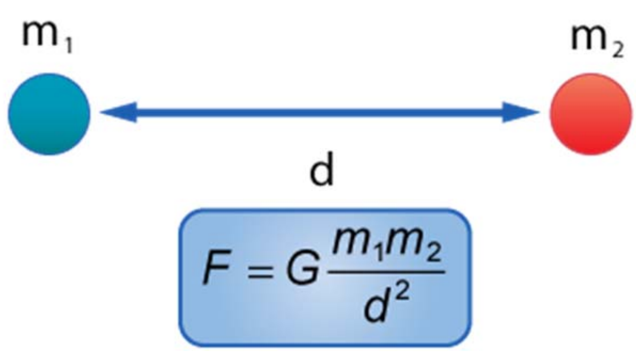
Begin this section by reading Chapter 1, Section 1.5, pp. 27 to 29 in the *Conceptual Physics* textbook.

View these videos from Kahn Academy on Introduction to Newton's Law of Gravity Part 1 and Gravity Part 2.

There are four fundamental forces in nature: (1) the gravitational force, (2) the electromagnetic force, (3) the strong nuclear force, and (4) the weak nuclear force. This section focuses on perhaps the most familiar, but weakest of these forces. When an object is dropped, it falls to the ground. Why? The answer to this question is very complex, and it has been the subject of numerous studies that have occupied the entire lives of some scientists for hundreds of years. Describing the force due to the gravitational attraction between objects is a much easier task, and explaining how this force operates is the subject of this section.

Consider the simplified case of two masses separated by a distance. The force of attraction between the two objects is proportional to the product of the masses and inversely proportional to the square of the distance between the objects. Inverse square laws are very common in nature. If a force has an inverse square relationship in regards to the distance between objects, then the distance between the objects increases by a factor of two (i.e., doubles) and the force decreases by a factor of two squared ($2^2 = 4$). Remember, when an increase in one variable causes the decrease in another variable (or vice versa), this is an inverse relationship.

Newton's Universal Law of Gravitation defines the force of attraction between any two masses in the universe. When the force of gravity is considered near the surface of the Earth, one of the masses is the mass of the Earth. A good approximation for the center of mass of the Earth is the geometric center of the Earth, and the distance between the objects is approximated by the radius of the Earth. The equation is usually represented as:



Note: G is a constant of proportionality that allows the force calculated to be expressed in Newtons if the masses are expressed in kilograms and the distance in meters.

The value for G is 6.67×10^{-11} N·m^2/kg^2.

Practice: Use Newton's Universal Law of Gravitation to fill in the missing values of F.

The correct answer has an up or down arrow with a number. The answers can be found at the end of the lesson.

Force increases(↑) or decreases (↓) by a factor of	m_1 increases (↑) or decreases (↓) by a factor of	m_2 increases(↑) or decreases (↓) by a factor of	d increases(↑) or decreases (↓) by a factor of
	↑2	1	1
	↑2	↑2	↑1
	↑2	↓2	↓2
	↑3	↑3	↑2
	↑3	↓2	↓4
	↓2	↓3	↑4

	↓3	↓3	↑5
	↑1	↑ 1	↑1.016

Weight is the measure of the force of gravity on an object. An object that is 100 km above the Earth's surface is considered to be in "outer space." The radius of the Earth is about 6,380 km. Therefore, outer space is not that much further than Earth's surface from the center of the Earth. The 1.016 in the last row of the table above represents the increase in distance for an object 100 km above Earth's surface. The force due to gravity only experienced a slight decrease. Why then are astronauts considered to be weightless in outer space if the force of gravity on them (i.e., their weight) is practically the same?

The next section examines projectile motion and satellite motion, a special case of projectile motion.

Section 5: Projectile and Satellite Motion

Begin this section by reading Chapter 10, Sections 10.1 and 10.2, pp. 234 to 241 in the *Newtonian Physics* textbook.

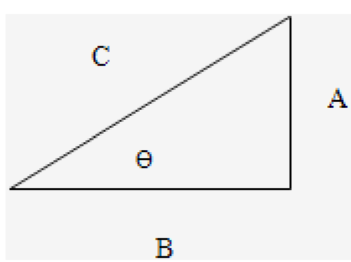
View these videos from Kahn Academy on Two-Dimensional Projectile Motion Part 1 and Part 2.

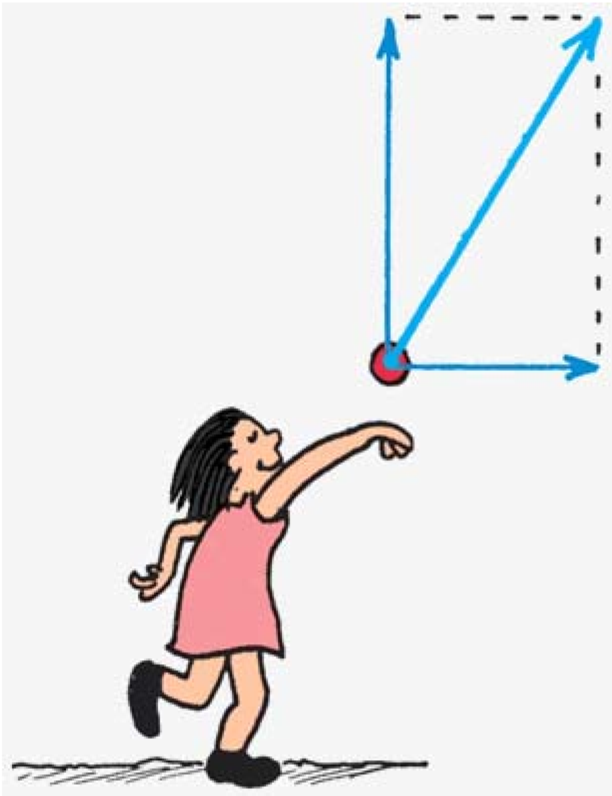
When a projectile is launched and the only force acting on the object is gravity, the object takes a parabolic path of travel.

Math Challenge:

The illustration below shows a girl throwing a ball with an initial velocity. The velocity is thrown at some angle with respect to the horizontal. This can be represented by a vector that has magnitude (amount) and direction. If the initial velocity and the angle of launch are known, the magnitudes of the horizontal and vertical components of the vector can easily be found.

In a right triangle, the sides of a right triangle are defined by the following relationships:





Copyright ©2007 Pearson Education, Inc., publishing as Pearson Addison-Wesley.

The sin Θ equals the ratio of A/C.

The cos Θ equals the ratio B/C.

The tan Θ equals the ratio B/A.

Θ is the angle shown in the diagram. Theta (Θ) is the angle of launch.

Values for the sin, cos, and tan of an angle can be found using a scientific calculator.

For the purposes of this section, a simple table is given below:

Angle Θ (degrees)	sin Θ	cos Θ	tan Θ
0	0	1	0
30	0.5	0.866	0.577
45	0.707	0.707	1
60	0.866	0.5	1.73
90	1	0	undefined

Therefore, in the illustration of the girl throwing the ball, if the ball is thrown at 20 m/s and the angle of launch is 60°, then the horizontal component can be found using the cos 60^o.

Cos 60^o = B/C or in other words, the side adjacent to the angle divided by the hypotenuse of the right triangle.

$$\text{Cos } 60^o = v_x/20 \text{ m/s}$$

where v_x is the horizontal component of the initial velocity.

Therefore $v_x = (20 \text{ m/s}) \cdot \cos 60^o = (20 \text{ m/s}) \cdot (0.5) = 10 \text{ m/s}$.

Likewise, $v_y = (20 \text{ m/s}) \cdot \sin 60^o = (20 \text{ m/s})\cdot(0.866) = 17.3 \text{ m/s}$

Check the math: Remember, for a right triangle (see the diagram above)

$$C^2 = A^2 + B^2 \text{ (Pythagorean Theorem)}$$

$$(20)^2 = (10)^2 + (17.3)^2$$

$$400 = 100 + 300$$

Knowing the horizontal and vertical components of the initial velocity allows you to calculate many variables associated with the projectile motion.

In the scenario above, the total time the projectile is in the air can be calculated. Use g =10 m/s^2 (instead of 9.8 m/s^2) to make the math easier. The time to reach the top of the path (where the vertical component of the velocity is zero) is found by dividing the initial velocity in the vertical direction by the acceleration in the vertical direction.

$$t = v_y/g$$

$$t = (17.3 \text{ m/s}) / (10 \text{ m/s}^2) = 1.73 \text{ s}$$

Since the time to the top of the path is the same as the time from the top of the path to the bottom, the total time the ball is in the air is 3.46 s. (round this off to 3.5 s).

In these 3.5 seconds, the ball was traveling at 10 m/s in the horizontal direction (no gravity is acting in the horizontal direction).

$$\text{Distance} = \text{Rate}\cdot\text{time} = 10 \text{ m/s}\cdot 3.5 \text{ s} = 35 \text{ m}$$

This is just a small sample of the variables that can be found.

Key Concept:

The variables that describe the motion, which are (1) distance traveled, (2) initial and final velocities, and (3) acceleration, can be separated into horizontal and vertical components and analyzed separately. The variable that connects the two motions is the time variable.

Math challenge:

In the scenario above, would the ball travel farther in the horizontal direction if the girl threw the ball at 20 m/s at an angle of 30^o instead of 60^o. Why or why not?

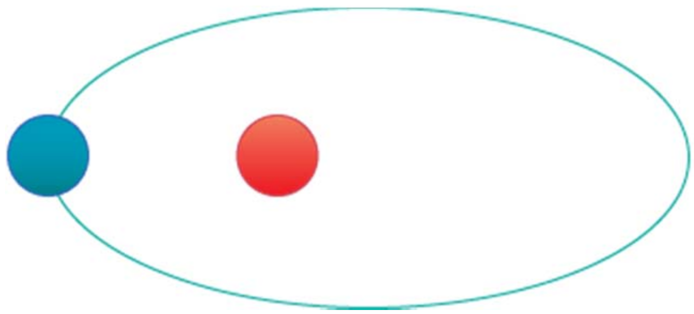
Answer - No, when the angles add up to °90 the horizontal range is the same. Total time in the air for the ball is now 2 s, however the initial horizontal velocity is now 17.3 m/s.

Distance = Rate·time = 17.3 m/s·2 s = 34.6 m = 35 m

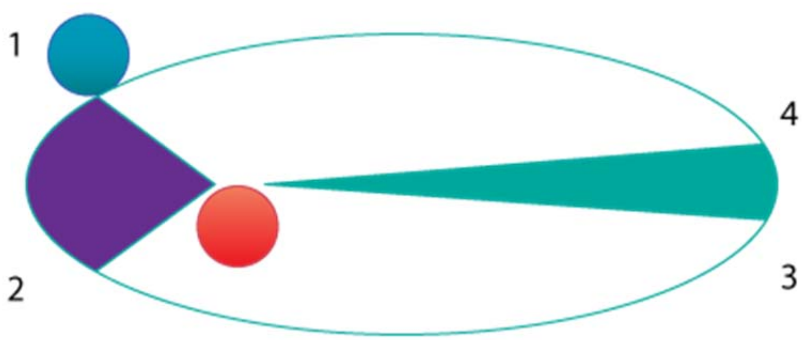
The next part of this section summarizes Kepler's three laws. Kepler's laws describe the motion of planets following an elliptical path around the Sun. Gravity is the force of attraction between the two bodies (planet and Sun) that allows this motion.

Kepler's Three Laws:

1. The paths of the planets are an ellipse, with the Sun at one focus.



2. An imaginary line drawn from the Sun to a planet sweeps out equal areas in equal time.



If the time to travel between points 1 and 2 is the same as the time to travel between points 3 and 4, the areas are the same.

Question: Is the Earth traveling faster or slower at position 1 compared to its speed at position 3, or is it possible that the speeds are equal?

Answer - THE EARTH IS TRAVELING FASTER AT POINT 1. It travels a greater distance in the same time, therefore it must be moving faster.

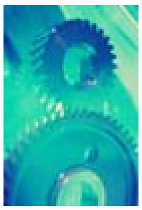
3. The square of the ratio of the periods of any two planets revolving about the Sun is equal to the cube of the ratio of their average distances from the sun.

Mathematically, this can be expressed as:

$$(T_1/T_2)^2 = (r_1/r_2)^3$$

Activity

Math challenge:



(Answers are located at the end of the lesson.)

- Uranus requires 84 years to circle the sun. Find Uranus's orbital radius as a multiple of Earth's orbital radius. Hint: Use Kepler's 3rd Law.
- Determine the mass of the sun given the Earth's distance from the sun as $d_{ES} = 1.5 \times 10^{11}$ m.

Summarizing Your Learning

This lesson focused on circular and rotational motion, which complement the linear motion studied earlier. Most motions in real-life examples are a combination of these motions. When real motion is analyzed, it is sometimes convenient to break the more complicated motion into the simple motions that make it up. On a basic scale, projectile motion is a prime example of this.

This lesson also discussed the gravitational force, which is the weakest of the four forces found in nature; however, its importance to the structure and operation of the universe cannot be overstated. The fundamental concepts addressed in the first five lessons provide the necessary foundation in physics to understand the remaining topics in the course.

Practice Answers

Check Prior Knowledge

Objects moving in a circle require a centrifugal force. FALSE

The gravitational force between two objects depends on the mass of the objects. TRUE

In order to escape the gravitational force of the Earth, a body must be launched at a specified minimum speed, regardless of the forces acting on the launched body. FALSE

Section 2: Angular Kinematics

Fill in the table shown below with the angular equation that corresponds to the linear equation given.

Linear	Angular
$s = s_o + v_o t + ½ at^2$	$\theta = \theta_o + \omega_o t + 1/2\, \alpha t^2$
$v = v_o + at$	$\omega = \omega_o + \alpha t$
$v^2 = v_o^2 + 2as$	$\omega^2 = \omega_o^2 + 2\alpha\theta$
$v_{avg} = (v_o + v_f)/2$	$\omega_{avg}\ (\omega_o + \omega_f)/2$

1. How long is an arc subtending 2.31 rad with a radius of curvature of 5 m?
   11.55 m

2. A disk spins at an angular velocity of -0.00804 rad/s. If it is an angle of 4.58 rad at time = 0, how much later will it be at an angle of 4.42 rad? 19.9 s

3. If a car accelerates from 34.4 km/h to 82.7 km/h in 12.5 s, and the tires have an angular acceleration of 5.339 rad/s^2, what is the radius of the tires? 21.1 cm

Section 4: Gravity

The correct answer has an up or down arrow with a number.

Force increases(↑) or decreases (↓) by a factor of	m_1 increases (↑) or decreases (↓) by a factor of	m_2 increases(↑) or decreases (↓) by a factor of	d increases(↑) or decreases (↓) by a factor of
↑2	↑2	1	1
↑4	↑2	↑2	↑1
↑4	↑2	↓2	↓2
↑3/2	↑3	↑3	↑2
↑96	↑3	↓2	↓4
↓3/8	↓2	↓3	↑4
↓9/25	↓3	↓3	↑5
↓0.9694	↑1	↑ 1	↑1.016

Section 5: Projectile and Satellite Motion

Math challenge:

- Uranus requires 84 years to circle the sun. Find Uranus's orbital radius as a multiple of Earth's orbital radius. Hint: Use Kepler's 3rd Law.

  Answer - Uranus is roughly 19 times further from the sun than the Earth is from the sun.

- Determine the mass of the sun given the Earth's distance from the sun as $d_{ES} = 1.5 \times 10^{11}$ m.

  Answer - 2.0×10^{30}